Gravity

Newtonian, Post-Newtonian, Relativistic

Eric Poisson and Clifford M. Will

CAMBRIDGE

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This textbook explores approximate solutions to general relativity and their consequences. It offers a unique presentation of Einstein's theory by developing powerful methods that can be applied to astrophysical systems.

Beginning with a uniquely thorough treatment of Newtonian gravity, the book develops post-Newtonian and post-Minkowskian approximation methods to obtain weak-field solutions to the Einstein field equations. The book explores the motion of self-gravitating bodies, the physics of gravitational waves, and the impact of radiative losses on gravitating systems. It concludes with a brief overview of alternative theories of gravity.

Ideal for graduate courses on general relativity and relativistic astrophysics, the book examines real-life applications, such as planetary motion around the Sun, the timing of binary pulsars, and gravitational waves emitted by binary black holes. Text boxes explore related topics and provide historical context, and over 100 exercises present challenging tests of the material covered in the main text.

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Preface

During the past forty years or so, spanning roughly our careers as teachers and research scientists, Einstein's theory of general relativity has made the transition from a largely mathematical curiosity with limited relevance to the real world to arguably the centerpiece of our effort to understand the universe on all scales.

At the largest scales, those of the universe as a whole, cosmology and general relativity are joined at the hip. You can't do one without the other. At the smallest scales, those of the Planck time, Planck length, and Planck energy, general relativity and particle physics are joined at the hip. String theory, loop quantum gravity, the multiverse, branes and bulk – these are arenas where the geometry of Einstein and the physics of the quantum may be inextricably linked. These days it seems that you can't do one without the other.

At the intermediate scales that interest astronomers, general relativity and astrophysics are becoming increasingly linked. You can still do one without the other, but it's becoming harder. One of us is old enough to remember a time when the majority of astronomers felt that black holes would never amount to much, and that it was a waste of time to worry about general relativity. Today black holes and neutron stars are everywhere in the astronomy literature, and gravitational lensing – the tool that relies on the relativistic bending of light – is used for everything from measuring dark energy to detecting exoplanets.

Given the surge of interest in general relativity, it is no surprise that the last several years have witnessed the publication of a multitude of new textbooks on Einstein's theory. Many of them are cut from a very similar cloth: they cover the fundamentals of the theory at an introductory level, including the spacetime formulation of special relativity, elements of differential geometry, the Einstein field equations, black holes, gravitational waves, and cosmology. This book is cut from a very different cloth. Here you will not (spoiler alert!) find any discussion of cosmology, and although black holes will appear in many places, you will not find anything about the joys and wonders of the Kerr metric.

This book is about *approximations* to Einstein's theory of general relativity, and their applications to planetary motion around the Sun, to the timing of binary pulsars, to gravitational waves emitted by binary black holes, and to many other real-life, astrophysical systems.

The first approximation to general relativity is, of course, Newton's gravity. Although the theories are conceptually very different, it must be admitted that the overwhelming majority of phenomena in the universe can be very adequately described by the laws of Newtonian gravity. To a high degree of accuracy, Newton rules the Sun, the Earth, the solar system, all normal stars, galaxies, and clusters of galaxies. Accordingly, almost a quarter of this book is devoted to Newton's theory. This choice reflects one of our (not so) hidden agendas. During our careers of teaching general relativity and advising graduate students, we have

too often encountered students who are superbly motivated to study Einstein's theory, but who cannot say more than "inverse square law" and "elliptical orbits" when asked what they know about Newtonian gravity. In our view, general relativity is a theory of gravity, and if you wish to comprehend its importance for astrophysics, you must first master what Newton has to say about gravitating bodies, rotating bodies, tidally interacting bodies, perturbed Keplerian orbits, and so on. We therefore make it our mission, in Chapters 1, 2, and 3, to provide a thorough discussion of the wonders of Newtonian gravity.

In the following two chapters we quickly review special relativity, the foundations of general relativity as a metric theory of gravity, the mathematical formulation of the theory, and its most famous solution, the Schwarzschild metric. We emphasize that Chapters 4 and 5 are very much a minimal package. The coverage is sufficient for our intended purposes in the remainder of the book, but it is no substitute for a proper education in general relativity that can be acquired from the traditional textbooks.

We get to our main point by Chapter 6. This is the development of a set of systematic schemes, known as post-Minkowskian theory and post-Newtonian theory, for obtaining approximate solutions to the Einstein field equations. The idea is to go from the exact theory, which governs the behavior of arbitrarily strong fields, such as those near black holes, to a useful approximation that applies to weak fields, such as those inside and near the Sun, those inside and near white dwarfs, and those at a safe distance from neutron stars and black holes. The approximation, of course, reproduces the predictions of Newtonian theory, but we go beyond this and formulate a method of approximation that can be pushed systematically to higher and higher order, and generate increasingly accurate descriptions of a weak gravitational field. Along the way, we make the case that this approximation can also describe important situations involving compact objects such as neutron stars and black holes; not the up-close-and-personal geometry of a compact object, to be sure, but its motion around another body (compact or not), so long as the mutual gravitational attraction is weak.

This program occupies us through Chapters 6, 7, 8, and 9. In Chapter 10 we apply the approximation methods to the description of relativistic effects on the dynamics of the solar system, the measurement of time on the Earth's surface and in orbit, the bending of light by a massive body, and the dynamics of spinning bodies. In Chapter 11 we explore the rich physics of gravitational waves, and in Chapter 12 we investigate the impact of radiative losses on the dynamics of gravitating systems. We conclude the book in Chapter 13 with a brief overview of alternative theories of gravity.

The central theme of this book is therefore the physics of weak gravitational fields. The reader may object that we give up too much by eliminating strong fields from our discussion; after all, exact solutions to the Einstein field equations describe the full richness of curved spacetime, whether strong or weak. Unfortunately, there are *extremely few* exact solutions to Einstein's equations that are physically interesting. The Schwarzschild solution is obviously interesting and important, and so is the Kerr solution for rotating black holes (although the Kerr metric makes no appearance in this book). But no exact solution to Einstein's equations has ever been found that describes a simple double-star system in orbital motion. And no exact solution is known that describes any kind of bounded, physical system that radiates gravitational waves.

The problem is that Einstein's field equations are so complicated that it is almost always necessary to impose a high degree of symmetry (spherical symmetry, spatial homogeneity, stationarity, etc.) in order to make progress toward finding a solution. Furthermore, a solution to Einstein's equations is, by definition, a *spacetime*; it must encompass the entire past history and future fate of the system, everywhere in space. For a binary-star system, for example, the solution must, at least in principle, run from the distant past, when a tenuous cloud of gas coalesced to form the stars, all the way to the distant future, when the stars, having possibly collapsed to form neutron stars or black holes along the way, have merged into a single object (possibly a single black hole); it must also describe the gravitational waves that are generated during the entire time by the orbital motion and merger of the two stars, and by the relaxation of the merged object to a final stationary state. It should not come as a surprise that nobody has found a solution that describes such a wide range of phenomena. Ironically, a body of beautiful mathematical work has demonstrated conclusively that given suitable initial conditions, a solution to Einstein's equations *always exists*, at least within a specified part of the spacetime. Sadly, such existence theorems do not tell us how to find such solutions.

Often, when one talks about exact solutions to the Einstein field equations, one means analytic solutions, or solutions that can be expressed in terms of reasonably well known mathematical functions. Perhaps this is too restrictive. What about numerical solutions? Given a sufficiently powerful computer, it should be possible to solve Einstein's equations numerically without imposing any symmetries. After all, the field equations of general relativity are partial differential equations, and these can readily be converted into the kind of difference equations that are suited to digital computing. This has turned out to be a very difficult challenge. Part of the difficulty is computational: simulation of the simplest spacetimes requires enormous computational power and memory. Part of the difficulty is mathematical: one must identify, from a broad spectrum of possibilities, a formulation of the field equations that is best suited for numerical work. There has been enormous progress on these fronts in the last 20 years, and spectacular breakthroughs have occurred in the last ten. Today (in 2013), numerical relativity is a major sub-branch of gravitational physics. It is now possible to simulate the final dozen orbits of two inspiralling and merging compact objects (black holes or neutron stars), the gravitational collapse of a dead stellar core on its way to form a supernova, the formation and evolution of accretion disks around black holes, the interaction of a binary neutron-star system with the strong magnetic fields it supports, and the generation of gravitational waves by such strongly gravitating systems.

As spectacular as this progress has been, at present it is still not possible to simulate the final thousand orbits of a compact binary inspiral. The limitations are both technical (a vast range of grid resolutions is required) and computational (insufficient memory and speed, even with the largest parallel processors). But approximately 990 of those orbits can be described by the weak-field methods that we develop in this book. It was found that there is a very good agreement between the approximation methods and those of numerical relativity when their domains of applicability overlap. So in addition to their obvious applications to the solar system, the weak-field methods have proved to be unreasonably effective in describing situations, such as the late stages of binary inspirals, where the fields are not so

weak and the motions not so slow. And the combination of these methods with numerical relativity has proved to be a powerful tool for many important problems.

The vast majority of high-precision experiments that were carried out to test general relativity can be fully understood on the basis of the post-Newtonian methods that we develop in this book. And even though the departures from Newtonian gravity are very, very small on and around Earth, modern technology has made them not only detectable, but also *essentially important* in the precision measurement of time. A well-known example is the Global Positioning System, which simply would not work if relativistic corrections were not taken into account. Today every relativist proudly points to the GPS as an example – admittedly, perhaps, the only example – of a practical application of general relativity. We describe how this comes about in Chapter 10.

Finally, a central motivation for this book is the expectation that soon after its initial publication, gravitational waves will be measured directly and routinely, and that gravitational-wave astronomy, enabled by ground-based laser interferometers, by pulsar timing arrays, and possibly by a future space-based antenna, will become a new standard way of "listening" to the universe. The approximation methods that we develop in this book are *the* tools for understanding gravitational radiation, and it is our hope that students and researchers wishing to join this new scientific venture will turn to our book to learn and master these tools.

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1 Foundations of Newtonian gravity

The central theme of this book is gravitation in its weak-field aspects, as described within the framework of Einstein's general theory of relativity. Because Newtonian gravity is recovered in the limit of very weak fields, it is an appropriate entry point into our discussion of weak-field gravitation. Newtonian gravity, therefore, will occupy us within this chapter, as well as the following two chapters.

There are, of course, many compelling reasons to begin a study of gravitation with a thorough review of the Newtonian theory; some of these are reviewed below in Sec. 1.1. The reason that compels us most of all is that although there is a vast literature on Newtonian gravity – a literature that has accumulated over more than 300 years – much of it is framed in old mathematical language that renders it virtually impenetrable to present-day students. This is quite unlike the situation encountered in current presentations of Maxwell's electrodynamics, which, thanks to books such as Jackson's influential text, are thoroughly modern. One of our main goals, therefore, is to submit the classical literature on Newtonian gravity to a Jacksonian treatment, to modernize it so as to make it accessible to present-day students. And what a payoff is awaiting these students! As we shall see in Chapters 2 and 3, Newtonian gravity is most generous in its consequences, delivering a whole variety of fascinating phenomena.

Another reason that compels us to review the Newtonian formulation of the laws of gravitation is that much of this material will be recycled and put to good use in later chapters of this book, in which we examine relativistic aspects of gravitation. Newtonian gravity, in this context, is a necessary warm-up exercise on the path to general relativity.

In this chapter we describe the foundations of the Newtonian theory, and leave the exploration of consequences to Chapters 2 and 3. We begin in Sec. 1.1 with a discussion of the domain of validity of the Newtonian theory. The main equations are displayed in Sec. 1.2 and derived systematically in Secs. 1.3 and 1.4. The gravitational fields of spherical and nearly-spherical bodies are described in Sec. 1.5, and in Sec. 1.6 we derive the equations that govern the center-of-mass motion of extended fluid bodies.

Gravitation rules the world, and before Einstein ruled gravitation, Newton was its king. In this chapter and the following two we pay tribute to the king.

1.1 Newtonian gravity

The gravitational theory of Newton is an extremely good representation of gravity for a host of situations of practical and astronomical interest. It accurately describes the structure

Table 1.1 Values of ε for representative gravitating systems.

Earth's orbit around the Sun	10^{-8}
Solar system's orbit around the galaxy	10^{-6}
Surface of the Sun	10^{-5}
Surface of a white dwarf	10^{-4}
Surface of a neutron star	0.1
Event horizon of a black hole	~ 1

of the Earth and the tides raised on it by the Moon and Sun. It gives a detailed account of the orbital motion of the Moon around the Earth, and of the planets around the Sun. To be sure, it is now well established that the Newtonian theory is not an exact description of the laws of gravitation. As early as the middle of the 19th century, observations of the orbit of Mercury revealed a discrepancy with the prediction of Newtonian gravity. This famous discrepancy in the rate of advance of Mercury's perihelion was resolved by taking into account the relativistic corrections of Einstein's theory of gravity. The high precision of modern measuring devices has made it possible to detect relativistic effects in the lunar orbit, and has made it necessary to take relativity into account in precise tracking of planets and spacecraft, as well as in accurate measurements of the positions of stars using techniques such as Very Long Baseline Radio Interferometry (VLBI). Even such mundane daily activities as using the Global Positioning System (GPS) to navigate your car in a strange city require incorporation of special and general relativistic effects on the observed rates of the orbiting atomic clocks that regulate the GPS network. But apart from these specialized situations requiring very high precision, Newtonian gravity rules the solar system.

Newtonian gravity also rules for the overwhelming majority of stars in the universe. The structure and evolution of the Sun and other main-sequence stars can be completely and accurately treated using Newtonian gravity. Only for extremely compact stellar objects, such as neutron stars and, of course, black holes, is general relativity important. Newtonian gravity is also perfectly capable of handling the structure and evolution of galaxies and clusters of galaxies. Even the evolution of the largest structures in the universe, the great galactic clusters, sheets and voids, whose formation is dominated by the gravitational influence of dark matter, are frequently modelled using numerical simulations based on Newton's theory, albeit with the overall expansion of the universe playing a significant role.

Generally speaking, the criterion that we use to decide whether to employ Newtonian gravity or general relativity is the magnitude of a quantity called the "relativistic correction factor" ε:

$$\varepsilon \sim \frac{GM}{c^2 r} \sim \frac{v^2}{c^2}\,, \tag{1.1}$$

where G is the Newtonian gravitational constant, c is the speed of light, and where M, r, and v represent the characteristic mass, separation or size, and velocity of the system under consideration. The smaller this factor, the better is Newtonian gravity as an approximation. Table 1.1 shows representative values of ε for various systems.

Context is everything, of course. It is now accepted that general relativity, not Newtonian theory, is the "correct" classical theory of gravitation. But in the appropriate context, Newton's theory may be completely adequate to do the job at hand to the precision required. For example, Table 1.1 implies that a description of planetary motion around the Sun, at a level of accuracy limited to (say) one part in a million, can safely be based on the Newtonian laws. The Newtonian theory can also be exploited to calculate the internal structure of white dwarfs, provided that one is content with a level of accuracy limited to one part in one thousand. For more compact objects, such as neutron stars and black holes, Newtonian theory is wholly inadequate.

1.2 Equations of Newtonian gravity

Most undergraduate textbooks begin their treatment of Newtonian gravity with Newton's second law and the inverse-square law of gravitation:

$$m_I \boldsymbol{a} = \boldsymbol{F}\,, \tag{1.2a}$$

$$\boldsymbol{F} = -\frac{G m_G M}{r^2}\boldsymbol{n}\,. \tag{1.2b}$$

In the first equation, $\boldsymbol{F}$ is the force acting on a body of inertial mass m_I situated at position $\boldsymbol{r}(t)$, and $\boldsymbol{a} = d^2\boldsymbol{r}/dt^2$ is its acceleration. In the second equation, the force is assumed to be gravitational in nature, and to originate from a gravitating mass situated at the origin of the coordinate system. The force law involves m_G, the passive gravitational mass of the first body at $\boldsymbol{r}$, while M is the active gravitational mass of the second body. The quantity G is Newton's constant of gravitation, equal to $6.6738 \pm 0.0008 \times 10^{-11}\ \mathrm{m^2\,kg^{-1}\,s^{-2}}$. The force is attractive, it varies inversely with the square of the distance $r := |\boldsymbol{r}| = (x^2 + y^2 + z^2)^{1/2}$, and it points in the direction opposite to the unit vector $\boldsymbol{n} := \boldsymbol{r}/r$. An alternative form of the force law is obtained by writing it as the gradient of a potential $U = GM/r$, so that

$$\boldsymbol{F} = m_G \nabla U\,. \tag{1.3}$$

This *Newtonian potential* will play a central role in virtually all chapters of this book.

If the inertial and passive gravitational masses of the body are equal to each other, $m_I = m_G$, then the acceleration of the body is given by $\boldsymbol{a} = \nabla U$, and its magnitude is $a = GM/r^2$. Under this condition the acceleration is independent of the mass of the body. This statement is known as the *weak equivalence principle* (WEP), and it was a central element in Einstein's thinking on his way to the concepts of curved spacetime and general relativity. Although Newton did not explicitly use our formulation in terms of inertial and passive masses, he was well aware of the significance of their equality. In fact, he regarded this equality as so fundamental that he opened his treatise *Philosophiae Naturalis Principia Mathematica* with a discussion of it; he even alluded to his own experiments showing that the periods of pendulums were independent of the mass and type of material suspended, which establishes the equality of inertial and passive masses (he referred to them

as the "quantity" and "weight" of bodies, respectively). Twentieth-century experiments have shown that the two types of mass are equal to parts in 10^{13} for a wide variety of materials (see Box 1.1).

Box 1.1 Tests of the weak equivalence principle

A useful way to discuss experimental tests of the weak equivalence principle is to parameterize the way it could be violated. In one parameterization, we imagine that a body is made up of atoms, and that the inertial mass m_I of an atom consists of the sum of all the mass and energy contributions of its constituents. But we suppose that the different forms of energy may contribute differently to the gravitational mass m_G than they do to m_I. One way to express this is to write

$$m_G = m_I(1 + \eta)\,,$$

where η is a dimensionless parameter that measures the difference. Because different forms of energy arising from the relevant subatomic interactions (such as electromagnetic and nuclear interactions) contribute different amounts to the total, depending on atomic structure, η could depend on the type of atom. For example, electrostatic energy of the nuclear protons contributes a much larger fraction of the total mass for high-*Z* atoms than for low-*Z* atoms.

Using this parameterization, we find from Eq. (1.2) that the acceleration of the body is given by

$$\boldsymbol{a} = -\frac{m_G}{m_I}\frac{GM}{r^2}\,\boldsymbol{n} = -\,(1+\eta)\,\frac{GM}{r^2}\,\boldsymbol{n}\,.$$

The difference in acceleration between two materials of different composition will then be given by

$$\Delta\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2 = -\,(\eta_1 - \eta_2)\,\frac{GM}{r^2}\,\boldsymbol{n}\,.$$

One way to place a bound on $\eta_1 - \eta_2$ is to drop two different objects in the Earth's gravitational field ($g = GM/r^2 \approx 9.8\,\mathrm{m\,s^{-2}}$), and compare their accelerations, or how long they take to fall. Although legend has it that Galileo Galilei verified the equivalence principle by dropping objects off the Leaning Tower of Pisa around 1590, in fact experiments like this had already been performed and were well known to Galileo; if he did indeed drop things off the Tower, he may simply have been performing a kind of classroom demonstration of an established fact for his students. Unfortunately, the "Galileo approach" is plagued by experimental errors, such as the difficulty of releasing the objects at exactly the same time, by the effects of air drag, and by the short time available for timing the drop.

A better approach is to balance the gravitational force (which depends on m_G) by a support force (which depends on m_I); the classic model is the pendulum experiments performed by Newton and reported in his *Principia*. The period of the pendulum depends on m_G/m_I, g, and the length of the pendulum. These experiments are also troubled by air drag, by errors in measuring or controlling the length of the pendulum, and by errors in timing the swing.

The best approach for laboratory tests was pioneered by Baron Roland von Eötvös, a Hungarian geophysicist working around the turn of the 20th century. He developed the *torsion balance*, schematically consisting of a rod suspended by a wire near its mid-point, with objects consisting of different materials attached at each end.